

ANALYTICITY ON FAMILIES OF CIRCLES

BY

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ABSTRACT

It is known that if f is a continuous function on the complex plane which extends holomorphically from each circle surrounding the origin, then f is not necessarily holomorphic. In the paper we prove that if, in addition, f extends holomorphically from each circle belonging to an open family of circles which do not surround the origin, then f is holomorphic.

1. Introduction and the main result

Write $\Delta(a, \rho) = \{\zeta \in \mathbb{C} : |\zeta - a| < \rho\}$ and $\Delta = \Delta(0, 1)$. If $0 < r_1 < r_2 < \infty$ write $A(a, r_1, r_2) = \{\zeta \in \mathbb{C} : r_1 \leq |\zeta - a| \leq r_2\}$. We say that a continuous function on $b\Delta(a, \rho)$ extends holomorphically from $b\Delta(a, \rho)$ if it has a continuous extension to $\overline{\Delta}(a, \rho)$ which is holomorphic on $\Delta(a, \rho)$.

A family \mathcal{C} of circles is called a **test family for holomorphy** (on \mathbb{C}) if every continuous function on \mathbb{C} that extends holomorphically from each circle in \mathcal{C} is holomorphic on \mathbb{C} . We will consider open families of circles, that is, families of the form $\{b\Delta(a, \rho) : (a, \rho) \in \mathcal{P}\}$ where \mathcal{P} is an open subset of $\mathbb{C} \times (0, \infty)$.

There are large families of circles that are not test families for holomorphy. For instance, the function

$$f(z) = \begin{cases} z^2/\bar{z} & (z \in \mathbb{C} \setminus \{0\}) \\ 0 & (z = 0) \end{cases}$$

is continuous on \mathbb{C} and extends holomorphically from each circle that surrounds the origin, yet f is not holomorphic. This shows that the family of all circles that surround the origin is not a test family for holomorphy. In the present paper we

prove that the family of all circles that surround the origin is a maximal open family that is not a test family for holomorphy:

THEOREM 1.1: *Let f be a continuous function on $\mathbb{C} \setminus \{0\}$ which extends holomorphically from each circle that surrounds the origin. Suppose that, in addition, f extends holomorphically from each circle belonging to a nonempty open family of circles that do not surround the origin. Then f is an entire function, that is, f is a holomorphic function on $\mathbb{C} \setminus \{0\}$ which has a removable singularity at 0.*

We prove Theorem 1.1 in the first part of the paper. In the second part we look at special cases of nonholomorphic continuous functions on $\mathbb{C} \setminus \{0\}$ which extend holomorphically from every circle surrounding the origin. In particular, we consider functions constant on lines passing through the origin and functions constant on rays passing through the origin.

To prove Theorem 1.1 we use a new approach to the holomorphic extension problem for circles which was introduced in [AG] and further developed in [G3]. We describe this new approach. In [AG] we studied rational functions of two real variables $f(z) = P(z, \bar{z})/Q(z, \bar{z})$ where P, Q are polynomials. We noticed that $f|_{b\Delta(a, \rho)}$ has a unique meromorphic extension to $\Delta(a, \rho)$ given by

$$(1.1) \quad f^*(z) = \frac{P(z, \bar{a} + \rho^2/(z - a))}{Q(z, \bar{a} + \rho^2/(z - a))},$$

so to say that f extends holomorphically from $b\Delta(a, \rho)$ means that f^* has no singularities in $\Delta(a, \rho)$. Given $a \in \mathbb{C}$ and $\rho > 0$ we introduced

$$\Lambda_{a, \rho} = \{(z, w) \in \mathbb{C}^2 : (z - a)(w - \bar{a}) = \rho^2, 0 < |z - a| < \rho\},$$

a closed complex submanifold of $\mathbb{C}^2 \setminus \Sigma$, attached to the real two-plane $\Sigma = \{(z, \bar{z}) : z \in \mathbb{C}\}$ along the circle $b\Lambda_{a, \rho} = \{(z, \bar{z}) : z \in b\Delta(a, \rho)\}$. The holomorphic extension of f from $b\Delta(a, \rho)$ to $\Delta(a, \rho)$ is then given as the restriction of the rational function $P(z, w)/Q(z, w)$ of two complex variables to the complex manifold $\Lambda_{a, \rho}$ as seen from (1.1).

In [G3] the varieties $\Lambda_{a, \rho}$ were used to formulate the holomorphic extension problem for general continuous functions as a problem in \mathbb{C}^2 , starting from the trivial observation that a continuous function f on $b\Delta(a, \rho)$ extends holomorphically to $\Delta(a, \rho)$ if and only if the function $F(z, \bar{z}) = f(z)$ defined on $b\Lambda_{a, \rho}$ has a bounded continuous extension to $\Lambda_{a, \rho} \cup b\Lambda_{a, \rho}$ which is holomorphic on $\Lambda_{a, \rho}$. If $A = A(b, r_1, r_2)$, it was shown that the union $\Omega(A)$ of all $\Lambda_{a, \rho}$ such that $b\Delta(a, \rho) \subset \text{Int } A$ surrounds b is a wedge domain attached to Σ along

$\tilde{A} = \{(z, \bar{z}): z \in A\}$. If f is a continuous function on A which extends holomorphically from each circle $b\Delta(a, \rho) \subset A$ surrounding b , then by an old result of the author [G1], the function f is the uniform limit of a sequence of polynomials in $z - b$ and $1/(\bar{z} - \bar{b})$ which implies that the function $F(z, \bar{z}) = f(z)$ ($z \in A$) has a bounded continuous extension to $\Omega(A) \cup b\Omega(A)$ which is holomorphic on $\Omega(A)$. So, roughly speaking, continuous functions f extendible holomorphically from open families of circles are the functions of the form $F(z, \bar{z})$ which are the boundary values of bounded holomorphic functions F on wedge domains attached to Σ . Clearly f is holomorphic if and only if F depends only on the first variable. At this point one can apply standard tools of several complex variables. We do this to prove Theorem 1.1.

A different formulation of the holomorphic extension problem as a problem in \mathbb{C}^2 has been used by A. Tumanov [T] for continuous functions f on the strip $\{z \in \mathbb{C}: |\Im z| \leq 1\}$ which extend holomorphically from each circle $b\Delta(t, 1)$, $t \in \mathbb{R}$. Tumanov defines a function F on $M = \{(\zeta + t, \zeta): \zeta \in \bar{\Delta}, t \in \mathbb{R}\}$, the disjoint union of translates of the disc $\{(\zeta, \zeta): \zeta \in \Delta\}$ in such a way that for each $t \in \mathbb{R}$, the function $\zeta \mapsto F(\zeta + t, \zeta)$ ($\zeta \in \bar{\Delta}$) is the continuous extension of $\zeta \mapsto f(\zeta + t)$ to $\bar{\Delta}$ which is holomorphic on Δ . In particular, $F(\zeta + t, \zeta) = f(\zeta + t)$ ($\zeta \in b\Delta, t \in \mathbb{R}$). He observes that since $F(z, w) = F(z, -1/w)$ ($w \in b\Delta$) one can extend F to $\tilde{M} = \{(\zeta + t, -1/\zeta): t \in \mathbb{R}, \zeta \in \bar{\Delta} \setminus \{0\}\}$ by $F(z, w) \equiv F(z, -1/w)$ to get a continuous CR function on the CR manifold $M \cup \tilde{M}$. He then constructs analytic discs attached to $M \cup \tilde{M}$ and uses the Baouendi–Treves approximation theorem, the edge of the wedge theorem and the continuity principle to prove that F does not depend on the second variable, that is, that f is holomorphic on $\{\zeta \in \mathbb{C}: |\Im \zeta| < 1\}$.

2. Varieties $\Lambda_{a,\rho}$ and domains $\Omega(A)$

Let $0 < r_1 < r_2 < \infty$ and let $a \in \mathbb{C}$. Denote by $\Omega(A(a, r_1, r_2))$ the union of all $\Lambda_{b,\rho}$ such that $b\Delta(b, \rho) \subset \text{Int } A(a, r_1, r_2)$ surrounds a . The set $\Omega(A(a, r_1, r_2))$ is an unbounded open connected subset of $\mathbb{C}^2 \setminus \Sigma$ which is attached to Σ along $\tilde{A}(a, r_1, r_2) = \{(z, \bar{z}): z \in A(a, r_1, r_2)\}$. We shall need the following

THEOREM 2.1 ([G3]): *Let f be a continuous function on $A(a, r_1, r_2)$. The following are equivalent:*

- (i) f extends holomorphically from each circle $b\Delta(b, \rho) \subset A(a, r_1, r_2)$ which surrounds the point a ;
- (ii) the function $F(z, \bar{z}) = f(z)$ defined on $\{(z, \bar{z}): z \in A(a, r_1, r_2)\}$ extends to a bounded continuous function on $\Omega(A(a, r_1, r_2)) \cup b\Omega(A(a, r_1, r_2))$ which is

holomorphic on $\Omega(A(a, r_1, r_2))$.

We list some simple properties of $\Lambda_{a,\rho}$ and $\Omega(A)$. The proofs are elementary. They can be found in [G3]. The proof of Proposition 2.1 can be found also in the earlier paper [AG].

PROPOSITION 2.1: *Let $(z, w) \in \mathbb{C}^2 \setminus \Sigma$. Then $(z, w) \in \Lambda_{a,R}$ if and only if there is a $t > 0$ such that $a = z + t(z - \bar{w})$ and $R = \sqrt{t(t+1)}|z - \bar{w}|$. In fact, given $R > 0$ we have*

$$(2.1) \quad a = z + 2^{-1}[\sqrt{1 + 4R^2/|z - \bar{w}|^2} - 1](z - \bar{w}).$$

Note that the two-dimensional subspace perpendicular to the Lagrangian two-plane Σ is $i\Sigma = \{(z, -\bar{z}): z \in \mathbb{C}\}$. Our next lemma tells how a variety $\Lambda_{a,\rho}$ intersects the two-dimensional planes perpendicular to Σ :

PROPOSITION 2.2: *Let $z \in \mathbb{C}, t > 0$ and $\varphi \in \mathbb{R}$. Then $(z, \bar{z}) + (te^{i\varphi}, -te^{-i\varphi}) \in \Lambda_{a,R}$ if and only if $a = z + \sqrt{t^2 + R^2}e^{i\varphi}$.*

PROPOSITION 2.3: *Let $A = A(a, r_1, r_2)$. Then $\Omega(A)$ is an unbounded open connected subset of $\mathbb{C}^2 \setminus \Sigma$ attached to Σ along $\{(z, \bar{z}): z \in \text{Int } A\}$. If $\gamma = (r_1 + r_2)/2$, then $b\Omega(A)$ consists of $\tilde{A} = \{(z, \bar{z}): z \in A\}$ together with all $\Lambda_{b,\rho}$ associated with those $b\Delta(b, \gamma) \subset A$ which are tangent to both $b\Delta(a, r_1)$ and $b\Delta(a, r_2)$. Further, $\Omega(A)$ is a disjoint union of $\Lambda_{b,\gamma}$ such that $b\Delta(b, \gamma) \subset \text{Int } A$.*

Let $z_0 \in \text{Int } A$. Let $\Gamma(z_0) \subset \text{Int } A$ be the circle of radius $\gamma = (r_1 + r_2)/2$ which passes through z_0 and whose center $b(z_0)$ lies on the line through a and z_0 . For each $\varphi \in \mathbb{R}$ define $T_\varphi(z) = z_0 + e^{i\varphi}(z - z_0)$. T_φ is the rotation with center z_0 for the angle φ . There is a $\delta(z_0), 0 < \delta(z_0) < \pi/2$, such that $T_\varphi(\Gamma(z_0)) \subset \text{Int } A (-\delta(z_0) < \varphi < \delta(z_0))$ and such that both $T_{\delta(z_0)}(\Gamma(z_0))$ and $T_{-\delta(z_0)}(\Gamma(z_0))$ meet bA (in fact, they meet both circles that bound A). Fix $\varphi, -\delta(z_0) < \varphi < \delta(z_0)$. There is a $\tau(z_0, \varphi) > 0$ such that

$$T_\varphi(\Gamma(z_0)) + t \frac{b(z_0) - z_0}{|b(z_0) - z_0|} e^{i\varphi} \subset \text{Int } A \quad (0 \leq t < \tau(z_0, \varphi))$$

while $T_\varphi(\Gamma(z_0)) + \tau(z_0, \varphi)((b(z_0) - z_0)/|b(z_0) - z_0|)e^{i\varphi}$ meets bA (in fact, it meets both circles that bound A). For each $\varphi, -\delta(z_0) < \varphi < \delta(z_0)$, let $\eta(z_0, \varphi) = \sqrt{\tau(z_0, \varphi)^2 + 2\tau(z_0, \varphi)\gamma}$. By Proposition 2.2 we have

$$(z_0, \bar{z}_0) + \left(t \frac{b(z_0) - z_0}{|b(z_0) - z_0|} e^{i\varphi}, -t \frac{\overline{b(z_0) - z_0}}{|b(z_0) - z_0|} e^{-i\varphi} \right) \in \Omega(A)$$

provided that $0 < t < \eta(z_0, \varphi)$. Let

$$D(z_0) = \left\{ t \frac{b(z_0) - z_0}{|b(z_0) - z_0|} e^{i\varphi} : 0 < t < \eta(z_0, \varphi), -\delta(z_0) < \varphi < \delta(z_0) \right\}.$$

It is easy to see that the function δ is continuous on $\text{Int } A$ and that η is a continuous function of z_0 and φ where it is defined. For each $z_0 \in \text{Int } A$ the set

$$(z_0, \bar{z}_0) + \{(\zeta, -\bar{\zeta}) : \zeta \in D(z_0)\}$$

is contained in $\Omega(A)$. This proves

PROPOSITION 2.4: *Let $z_0 \in \text{Int } A$. There are a neighbourhood $U \subset \Sigma$ of (z_0, \bar{z}_0) , an open convex cone $K \subset \mathbb{C}$ with vertex at the origin, containing $\{t(a - z_0) : t > 0\}$, and an $r > 0$ such that if*

$$P = \{(\zeta, -\bar{\zeta}) : \zeta \in K, |\zeta| < r\},$$

then $U + P \subset \Omega(A)$.

Fix $R > 0$. From (2.1) we get that if $(z, w) \in \Lambda_{a,R}$ then $|a| \leq |z| + 2R^2/|z - \bar{w}|$ which, by Proposition 2.3, implies that given $r_1, r_2, 0 < r_1 < r_2 < \infty$,

$$(2.2) \quad \begin{aligned} &\text{there are } \delta > 0 \text{ and } M < \infty \text{ such that} \\ &\{(z, w) : |z| \leq \delta, |w| \geq M\} \subset \Omega(A(0, r_1, r_2)). \end{aligned}$$

3. Functions that extend holomorphically from every circle which surrounds the origin

Suppose that f is a continuous function on $\mathbb{C} \setminus \{0\}$ which extends holomorphically from each circle that surrounds the origin. Define F on $\Sigma \setminus \{(0, 0)\}$ by

$$F(z, \bar{z}) = f(z).$$

Then for each a, ρ such that $b\Delta(a, \rho)$ surrounds the origin, the function $F|_{b\Lambda_{a,\rho}}$ has a bounded continuous extension to $\Lambda_{a,\rho} \cup b\Lambda_{a,\rho}$ which is holomorphic on $\Lambda_{a,\rho}$. By Theorem 2.1 we know that this defines a holomorphic function F on Ω , the union of all $\Lambda_{a,\rho}$ such that $b\Delta(a, \rho)$ surrounds the origin. By Theorem 2.1 for each $r_1, r_2, 0 < r_1 < r_2 < \infty$, the restriction of F to $\Omega(A(0, r_1, r_2))$ is bounded and has a (bounded) continuous extension to $\Omega(A(0, r_1, r_2)) \cup b\Omega(A(0, r_1, r_2))$ which, on $\tilde{A}(0, r_1, r_2) = \{(z, \bar{z}) : \zeta \in A(0, r_1, r_2)\}$, coincides with $F(z, \bar{z})$.

We now show that

$$\Omega = \{(z, w) : |w| > |z|\}.$$

One way to see this is by using Proposition 2.2. We show this by using Proposition 2.1.

Suppose that $(z, w) \in \mathbb{C}^2 \setminus \Sigma$, that is, $w \neq \bar{z}$. By Proposition 2.1 we have $(z, w) \in \Lambda_{a,\rho}$ if and only if

$$(3.1) \quad a = z + t(z - \bar{w}), \rho = \sqrt{t(t+1)}|z - \bar{w}|$$

for some $t > 0$. Let L be the line through $(z + \bar{w})/2$ which is perpendicular to the line through z and \bar{w} and let $\Pi \subset \mathbb{C}$ be the open halfplane bounded by L which contains z . It is easy to see that Π is the union of all $\Delta(a, \rho)$ such that a and ρ satisfy (3.1) for some $t > 0$. It follows that $(z, w) \in \Lambda_{a,\rho}$ for some $b\Delta(a, \rho)$ that surrounds the origin if and only if $0 \in \Pi$, that is, if and only if $|w| > |z|$.

For each $z \neq 0$ we describe $[(z, \bar{z}) + i\Sigma] \cap \Omega$. Recall that $i\Sigma = \{(\zeta, -\bar{\zeta}) : \zeta \in \mathbb{C}\}$. Write $z = |z|e^{i\alpha}$. Then $(z, \bar{z}) + (\zeta, -\bar{\zeta}) \in \Omega$ if and only if $|z + \zeta| < |\bar{z} - \bar{\zeta}|$, that is, if and only if $\operatorname{Re}(\bar{z}\zeta) < 0$. This happens if and only if $\operatorname{Re}(e^{-i\alpha}\zeta) < 0$, that is, if and only if $\zeta \in e^{i\alpha}\{z : \operatorname{Re} z < 0\}$. Let $L(z)$ be the line through the origin which is perpendicular to the line through 0 and z and let $P(z)$ be the halfplane bounded by $L(z)$ which does not contain z . Then

$$[(z, \bar{z}) + i\Sigma] \cap \Omega = (z, \bar{z}) + \{(\zeta, -\bar{\zeta}) : \zeta \in P(z)\}.$$

Obviously $i\Sigma \cap \Omega = \emptyset$. Thus, Ω can be written as a disjoint union of halfplanes

$$\Omega = \bigcup_{z \in \mathbb{C} \setminus \{0\}} [(z, \bar{z}) + \{(\zeta, -\bar{\zeta}) : \zeta \in P(z)\}].$$

Further,

$$b\Omega = (i\Sigma) \cup \left(\bigcup_{z \in \mathbb{C} \setminus \{0\}} \{(\zeta, -\bar{\zeta}) : \zeta \in L(z)\} \right).$$

Note that we cannot conclude in general that the function F extends continuously to $\Sigma \setminus \{(0, 0)\}$. If $(z, \bar{z}) \in \Sigma \setminus \{(0, 0)\}$ and if $(z_n, w_n) \in \Omega$, $(z_n, w_n) \rightarrow (z, \bar{z})$, then $\lim_{n \rightarrow \infty} F(z_n, w_n) = F(z, \bar{z}) = f(z)$ provided that there are $r_1, r_2, 0 < r_1 < r_2 < \infty$ such that $(z_n, w_n) \in \Omega(A(0, r_1, r_2))$ for all n . However, we have the following

PROPOSITION 3.1: *Let f and F be as above. Suppose that $F(z, \bar{z}) = f(z)$ has a holomorphic extension Φ into an open ball $B \subset \mathbb{C}^2 \setminus \{(0, 0)\}$ centered at $(z_0, \bar{z}_0) \in \Sigma \setminus \{(0, 0)\}$. Then $\Phi \equiv F$ on $B \cap \Omega$.*

Proof: By Proposition 2.4 there are a neighbourhood $U \subset \Sigma$ of (z_0, \bar{z}_0) , an open convex cone $K \subset i\Sigma$ with vertex at the origin and an $\eta > 0$ such that if

$K_\eta = \{w \in K, |w| < \eta\}$ then $U + K_\eta \subset \Omega(A(0, r_1, r_2) \cap B$ for some $r_1, r_2, 0 < r_1 < r_2 < \infty$ and hence $F(z, \bar{z})$ has a continuous extension from U to $U \cup [U + K_\eta]$ which is holomorphic on $U + K_\eta$. However, such extension is unique and since $\Phi[[U \cup [U + K_\eta]]]$ is such an extension we must have $\Phi \equiv F$ on $U + K_\eta$. Since $(U + K_\eta) \cap B$ is an open subset of $\Omega \cap B$ and since $\Omega \cap B$ is connected, it follows that $\Phi \equiv F$ on $\Omega \cap B$. This completes the proof.

4. Intersecting varieties $V_{a,\rho}$ with Ω

Given $a \in \mathbb{C}$ and $\rho > 0$ let

$$V_{a,\rho} = \{(z, w) : (z - a)(w - \bar{a}) = \rho^2\}.$$

Thus, $\Lambda_{a,\rho} = \{(z, w) \in V_{a,\rho} : 0 < |z - a| < \rho\}$ is one of the two components of $V_{a,\rho} \setminus b\Lambda_{a,\rho}$. We compute $V_{a,\rho} \cap b\Omega$. The equation of $V_{a,\rho}$ is $w = \bar{a} + \rho^2/(z - a)$, so we compute the intersection of $V_{a,\rho}$ with $b\Omega = \{(z, w) : |z| = |w|\}$ by solving $|\bar{a} + \rho^2/(z - a)| = |z|$. We get $|\bar{a}(z - a) + \rho^2| = |z| \cdot |z - a|$ so

$$[\bar{a}(z - a) + \rho^2] \cdot [a(\bar{z} - \bar{a}) + \rho^2] - \rho^2 z \bar{z} = z \bar{z} = z \bar{z} [(z - a)(\bar{z} - \bar{a}) - \rho^2].$$

The left-hand side equals

$$\begin{aligned} & a\bar{a}[(z - a) + \rho^2/\bar{a}] \cdot [(\bar{z} - \bar{a}) + \rho^2/a] - \rho^2[(z - a)(\bar{z} - \bar{a}) + a\bar{z} + \bar{a}z - a\bar{a}] \\ &= a\bar{a}(z - a)(\bar{z} - \bar{a}) + \rho^4 - \rho^2(z - a)(\bar{z} - \bar{a}) + a\bar{a}\rho^2 \\ &= (a\bar{a} - \rho^2)[(z - a)(\bar{z} - \bar{a}) - \rho^2] \end{aligned}$$

and the equation becomes

$$[z\bar{z} - (a\bar{a} - \rho^2)] \cdot [(z - a)(\bar{z} - \bar{a}) - \rho^2] = 0.$$

If the circle $b\Delta(a, \rho)$ surrounds the origin, that is, if $|a| < \rho$, then the set of solutions is $b\Delta(a, \rho)$. The case of interest to us will be the case when $|a| > \rho$, that is, when $b\Delta(a, \rho)$ does not surround the origin. In this case the set of solutions is $b\Delta(a, \rho) \cup b\Delta(0, \sqrt{|a|^2 - \rho^2})$. Note that these two circles intersect at right angle.

Since

$$(4.1) \quad |z - a|^2 [|\bar{a} + \rho^2/(z - a)|^2 - |z|^2] = [(a\bar{a} - \rho^2 - z\bar{z}) \cdot [(z - a)(\bar{z} - \bar{a}) - \rho^2]]$$

the point $(z, \bar{a} + \rho^2/(z - a))$ belongs to Ω if and only if the expression on the left in (4.1) is positive, that is, if and only if the expression on the right in (4.1)

is positive. This happens if and only if either $z \in \Delta(0, \sqrt{|a|^2 - r^2}) \setminus \overline{\Delta}(a, r)$ or $z \in \Delta(a, r) \setminus \overline{\Delta}(0, \sqrt{|a|^2 - r^2})$. Thus, if

$$\begin{aligned} D_1(a, r) &= \Delta(a, r) \cap \Delta(0, \sqrt{|a|^2 - r^2}), \\ D_2(a, r) &= \mathbb{C} \setminus [\overline{\Delta}(a, r) \cup \overline{\Delta}(0, \sqrt{|a|^2 - r^2})], \\ D_3(a, r) &= \Delta(0, \sqrt{|a|^2 - r^2}) \setminus \overline{\Delta}(a, r), \\ D_4(a, r) &= \Delta(a, r) \setminus \overline{\Delta}(0, \sqrt{|a|^2 - r^2}), \end{aligned}$$

and

$$\begin{aligned} V_i(a, r) &= \{(z, \bar{a} + \rho^2/(z - a)) : z \in D_i(a, r)\} (1 \leq i \leq 3), \\ V_4(A, r) &= \{(z, \bar{a} + \rho^2/(z - a)) : z \in D_4(a, r) \setminus \{a\}\}, \end{aligned}$$

then $V_1(a, r)$ and $V_2(a, r)$ are the components of $V_{a,r} \setminus \overline{\Omega}$ and $V_3(a, r)$ and $V_4(a, r)$ are the components of $V_{a,r} \cap \Omega$.

5. Outline of the proof of Theorem 1.1

We start with a continuous function f on $\mathbb{C} \setminus \{0\}$ which extends holomorphically from every circle which surrounds the origin and the associated function F , holomorphic on $\Omega = \{(z, w) : |w| > |z|\}$. Suppose that f extends holomorphically from a circle $b\Delta(b, r)$ that does not surround the origin and from all nearby circles $b\Delta(a, r)$ with a close to b . Then $F|_{b\Lambda_{b,r}}$ has a bounded continuous extension F_1 to $\Lambda_{b,r} \cup b\Lambda_{b,r}$ which is holomorphic on $\Lambda_{b,r}$, in particular, F_1 on $\Lambda_{b,r} \cap \Omega = V_4(b, r)$. We use the edge of the wedge theorem as in [G3] to show that, since f extends holomorphically from all nearby circles $b\Delta(a, \rho)$, on $V_4(b, r)$, the function $F_1|_{V_4(b, r)}$ coincides with $F|_{V_4(b, r)}$. Since F_1 is holomorphic on $\Lambda_{a,r}$, it follows that F extends holomorphically along $V_{b,r}$ to $V_1(b, r)$. Further, using again the fact that f extends holomorphically from each circle $b\Delta(a, r)$ where a runs through a neighbourhood of b and repeating the process above with b replaced by a , we see that F extends holomorphically into a neighbourhood P of $\overline{V_1(b, r)}$ in \mathbb{C}^2 . Now we can apply the continuity principle. $V_1(b, r)$ is a holomorphically embedded disc which can be continuously deformed through a family of holomorphically embedded discs into a holomorphically embedded disc lying on the w -axis which contains the origin in its interior, in such a way that boundaries of all these discs are contained in $\Omega \cup P$. This implies that F extends holomorphically into a neighbourhood of the origin. This shows that f can be defined at 0 so that it becomes a real analytic function in a neighbourhood of the origin. By a result from [G2] it follows that f is holomorphic on \mathbb{C} , which will complete the proof.

6. Proof of Theorem 1.1

Suppose that f extends holomorphically from each circle that surrounds the origin. We know that there is a holomorphic function F on Ω which, for each $R_1, R_2, 0 < R_1 < R_2$, has a bounded continuous extension to $\Omega(A(0, R_1, R_2)) \cup b\Omega(A(0, R_1, R_2))$ which coincides with $F(z, \bar{z}) = f(z)$ on $\tilde{A}(0, R_1, R_2) = \{(z, \bar{z}): z \in A(0, R_1, R_2)\}$.

Suppose that $b \in \mathbb{C}, 0 < r_1 < r_2 < |b|$ and suppose that f extends holomorphically from each circle $b\Delta(a, \rho) \subset A(b, r_1, r_2)$ which surrounds b . Notice that no such $b\Delta(a, \rho)$ surrounds the origin. By Theorem 2.1 the function $F(z, \bar{z}) = f(z)$ has a bounded continuous extension F_1 from $\tilde{A}((b, r_1, r_2)$ to $\Omega(A(b, r_1, r_2)) \cup b\Omega(A(b, r_1, r_2))$ which is holomorphic on $\Omega(A(b, r_1, r_2))$.

Let $r = (r_1 + r_2)/2$ and consider the circle $b\Delta(b, r)$ and associated varieties $\Lambda_{b,r}$ and $V_{b,r}$. Note that $\Omega(A(b, r_1, r_2))$ is an open neighbourhood of $\Lambda_{b,r}$. Recall that $\Omega(A(b, r_1, r_2))$ is the disjoint union of $\Lambda_{a,r}$ such that $b\Delta(a, r) \subset \text{Int } A(b, r_1, r_2)$ surrounds b .

Write $V_j = V_j(b, r), 1 \leq j \leq 4$ and let

$$\lambda = \{(z, \bar{b} + r^2/(z - b)): z \in b\Delta(b, r) \cap \bar{\Delta}(0, \sqrt{|b|^2 - r^2})\}.$$

Note that λ is an arc which is a part of $b\Lambda_{b,r} \subset b\Omega$. Clearly

$$w \in b\Delta(b, r) \cap \bar{\Delta}(0, \sqrt{|b|^2 - r^2}) \quad \text{if and only if } (w, \bar{w}) \in \lambda.$$

Each such w is contained in two circles, tangent from outside to each other at w , one contained in $\text{Int } A(0, R_1, R_2)$ for some $R_1, R_2, 0 < R_1 < R_2 < \infty$, and surrounding the origin, and the other contained in $\text{Int } A(b, r_1, r_2)$ and surrounding b . Proposition 2.4 implies that there are a neighbourhood $U \subset \Sigma$ of (w, \bar{w}) , an open convex cone $K \subset i\Sigma$ with vertex at the origin, and an $\eta > 0$ such that if $K_\eta = \{Z \in K, |Z| < \eta\}$ then $U + K_\eta \subset \Omega(A(0, R_1, R_2))$ and $U - K_\eta \subset \Omega(A(b, r_1, r_2))$. By the edge of the wedge theorem it follows that $F(z, \bar{z}) = f(z)$ has a holomorphic extension Φ into a small open ball $B \subset \mathbb{C}^2$ centered at (w, \bar{w}) . Provided that B is small enough Proposition 3.1 implies that $\Phi \equiv F$ on $B \cap \Omega$ and $\Phi \equiv F_1$ on $B \cap \Omega(A(b, r_1, r_2))$. Since we can repeat the process for every $(w, \bar{w}) \in \lambda$ it follows that there is an open connected neighbourhood \mathcal{V} of λ in \mathbb{C}^2 such that $\mathcal{V} \cap \Omega, \mathcal{V} \cap \Omega(A(b, r_1, r_2))$ and $\mathcal{V} \cap \tilde{A}(b, r_1, r_2)$ are connected, such that $F(z, \bar{z}) = f(z)$ has a holomorphic extension Φ to \mathcal{V} which satisfies $\Phi \equiv F$ on $\Omega \cap \mathcal{V}$ and $\Phi \equiv F_1$ on $\Omega(A(b, r_1, r_2)) \cap \mathcal{V}$.

The components V_3 and V_4 of $V_{b,r} \cap \Omega$ are contained in Ω , so F is well defined and holomorphic on V_3 and V_4 . The function F_1 is well defined on components

V_1 and V_4 of $V_{b,r} \setminus b\Omega$ which, together with the arc

$$\{(z, \bar{b} + r^2/(z - b)): z \in b\Delta(0, \sqrt{|b|^2 - r^2}) \cap \Delta(b, r)\},$$

form $\Lambda_{b,r}$. We first show that on V_4 , where both F and F_1 are defined, these two functions coincide. To see this, choose $w \in b\Delta(b, r) \setminus \overline{\Delta(0, \sqrt{|b|^2 - r^2})}$. There is a disc $\Delta(c, R)$ which contains the origin such that $\Delta(b, r) \subset \Delta(c, R)$ and such that $b\Delta(c, R)$ is tangent to $b\Delta(b, r)$ at w . Proposition 2.4 implies that there are an open neighbourhood $U \subset \Sigma$ of (w, \bar{w}) , an open convex cone $K \subset i\Sigma$ with vertex at the origin and an $\eta > 0$ such that if $K_\eta = \{Z \in K, |Z| < \eta\}$ then $U + K_\eta \subset \Omega(A(b, r_1, r_2)) \cap \Omega(A(0, R_1, R_2))$ for some $R_1, R_2, 0 < R_1 < R_2 < \infty$ and such that V_4 meets $U + K_\eta$. This implies that $F \equiv F_1$ on $U + K_\eta$ since their boundary values $F(z, \bar{z}) = f(z) = F_1(z, \bar{z}) ((z, \bar{z}) \in U)$ are the same. Since V_4 meets $U + K_\eta$ it follows that $F \equiv F_1$ on V_4 .

The arc λ is the intersection of the boundaries of V_1 and V_3 in $V_{b,r}$. We show that $F_1|_{V_1}$ is the analytic continuation of $F|_{V_3}$ in $V_{b,r}$ across $\text{Int } \lambda$. To see this, recall that there are an open neighbourhood $\mathcal{V} \subset \mathbb{C}^2$ of λ and a holomorphic function Φ on \mathcal{V} such that $\Phi \equiv F$ on $\mathcal{V} \cap \Omega$ and $\Phi \equiv F_1$ on $\mathcal{V} \cap \Omega(A(b, r_1, r_2))$, so there is a single holomorphic function $\Psi = \Phi|_{V_{b,r} \cap \mathcal{V}}$ on $V_{b,r} \cap \mathcal{V}$ such that $\Psi \equiv F$ on $V_3 \cap \mathcal{V}$ and $\Psi \equiv F_1$ on $V_1 \cap \mathcal{V}$.

Thus we showed that F extends holomorphically into a neighbourhood \mathcal{V} of λ in \mathbb{C}^2 , that $F|_{V_3 \cup V_4}$ extends holomorphically along $V_{b,r}$ into a neighbourhood of $\overline{V_1}$ in $V_{b,r}$ and that $F|_{V_4} \equiv F_1|_{V_4}$.

We now use the preceding reasoning further to show that F extends holomorphically to a neighbourhood of $\overline{V_1}$ in \mathbb{C}^2 . To see this, we choose a small $\eta > 0$ and repeat the process above with $V_{a,r}, a \in \Delta(b, r)$, in place of $V_{b,r}$. Note that the union \mathcal{W} of all $\Lambda_{a,r}, a \in \Delta(b, r)$, is an open neighbourhood of $\Lambda_{b,r}$ which, provided that η is small enough, is contained in $\Omega(A(b, r_1, r_2))$ and so F_1 is holomorphic on \mathcal{W} . Note that $\mathcal{V} \cup \mathcal{W}$ is a neighbourhood of $\overline{V_1}$ in \mathbb{C}^2 . Repeating the process above for $a \in \Delta(b, \eta)$ in place of b we see that $F|_{V_3(a, r) \cup V_4(a, r)}$ extends holomorphically along $V_{a,r}$ into a neighbourhood of $\overline{V_1(a, r)}$ in $V_{a,r}$. However, in \mathcal{W} all these extensions coincide with F_1 so if the function Ψ on $\mathcal{V} \cup \mathcal{W}$ is defined as $\Psi|_{\mathcal{V}} = F|_{\mathcal{V}}$ and $\Psi|_{\mathcal{W}} = F_1|_{\mathcal{W}}$, then Ψ is holomorphic on $\mathcal{V} \cap \mathcal{W}$. Thus, F extends holomorphically into $\mathcal{V} \cap \mathcal{W}$, a neighbourhood of $\overline{V_1}$ in \mathbb{C}^2 .

We will now apply the continuity principle. Recall that F extends holomorphically into a neighbourhood P of $\overline{V_1}$ in \mathbb{C}^2 . Now,

$$V_1 = \{(z, \bar{b} + r^2/(z - b)): z \in D_1(b, r)\}$$

is an embedded analytic disc whose boundary

$$bV_1 = \{(z, \bar{b} + r^2/(z - b)) : z \in bD_1(b, r)\}$$

is contained in $b\Omega$. For each $t, 0 \leq t \leq 1$, let

$$V_{1,t} = \{(tz, \bar{b} + r^2/(z - b)) : z \in D_1(b, r)\}.$$

Then $V_{1,t}, 0 \leq t \leq 1$, is a continuous family of embedded analytic discs, $V_{1,1} = V_1$, whose boundaries

$$bV_{1,t} = \{(tz, \bar{b} + r^2/(z - b)) : z \in bD_1(b, r)\}$$

are contained in $\Omega \cup P$ (in fact, for $0 \leq t < 1$ they are contained in Ω). By the continuity principle it follows that F extends holomorphically into a neighbourhood of

$$\overline{V_{1,0}} = \{(0, \bar{b} + r^2/(z - b)) : z \in \overline{D_1(b, r)}\}$$

in \mathbb{C}^2 . It is easy to see that $V_{1,0}$ contains the origin. Consequently, F extends holomorphically into a neighbourhood of the origin in \mathbb{C}^2 and so f extends across the origin in \mathbb{C} as a function which is real analytic in a neighbourhood of the origin. Since f extends holomorphically from every circle surrounding the origin it follows from [G2] that f is holomorphic on \mathbb{C} . This completes the proof.

Remark: In the last step of the proof above we may, instead of [G2], use the Liouville theorem as follows: From (2.2) it follows that F is bounded on $\Delta(0, \delta) \times \mathbb{C}$ for some $\delta > 0$. By the Liouville theorem the function $\zeta \mapsto F(z, \zeta)$ is constant for each $z \in \Delta(0, \delta)$ so F does not depend on w on $\Delta(0, \delta) \times \mathbb{C}$. It follows that F is a function of z only so $f(z) = F(z, \bar{z})(z \in \mathbb{C} \setminus \{0\})$ is a restriction of an entire function to $\mathbb{C} \setminus \{0\}$.

7. Examples

By Theorem 1.1 the family of all circles that surround the origin is a maximal open family of circles that is not a test family for holomorphy. Even its closure, that is the family of all circles that either surround the origin or pass through the origin, is not a maximal family that is not a test family for holomorphy. To see this, let $a \in \mathbb{C}, \rho > 0, |a| > \rho$, and let

$$g(z) = \begin{cases} 0 & \text{if } z = 0, \\ (z^2/\bar{z})[(z - a)(\bar{z} - \bar{a}) - \rho^2] & \text{if } z \neq 0. \end{cases}$$

The function g vanishes identically on $b\Delta(a, \rho)$ and hence extends holomorphically from $b\Delta(a, \rho)$. Since

$$(z^2/\bar{z})[(z - a)(\bar{z} - \bar{a}) - \rho^2] = z^3 - az^2 - \bar{a}(z^3/\bar{z}) + a\bar{a}(z^2/\bar{z}) - \rho^2(z^2/\bar{z})$$

is a polynomial in z and $1/\bar{z}$ it follows that g extends holomorphically from every circle that surrounds the origin. Since g is continuous on \mathbb{C} it extends holomorphically also from every circle that passes through the origin. This shows that if $|a_i| > \rho_i > 0, 1 \leq i \leq n$, then the function

$$f(z) = \begin{cases} 0 & \text{if } z = 0 \\ (z^2/z)^n \prod_{j=1}^n [(z - a_j)(\bar{z} - \bar{a}_j) - \rho_j^2] & \text{if } z \neq 0 \end{cases}$$

is continuous on \mathbb{C} , extends holomorphically from all circles that surround the origin, from all circles that pass through the origin, and from all circles $b\Delta(a_i, \rho_i), 1 \leq i \leq n$, yet f is not holomorphic.

In our next example, let g be a function from the disc algebra and define

$$(7.1) \quad f(z) = g(z/\bar{z}) \quad (z \in \mathbb{C} \setminus \{0\}).$$

Suppose that $b\Delta(a, \rho)$ surrounds the origin. Then $|a| < \rho$ and for $|\zeta| = 1$ we have

$$f(a + \zeta\rho) = g\left(\frac{a + \zeta\rho}{\bar{a} + \bar{\zeta}\rho}\right) = g\left(\zeta \frac{\zeta + a/\rho}{1 + (\bar{a}/\rho)\zeta}\right),$$

which shows that the function $\zeta \mapsto f(a + \zeta\rho)$ ($\zeta \in b\Delta$) extends to a function from the disc algebra. Thus, f extends holomorphically from every circle surrounding the origin. Since the boundary values of the functions from the disc algebra can be highly non-smooth, this example shows that a highly non-smooth function on $\mathbb{C} \setminus \{0\}$ can be holomorphically extendible from every circle surrounding the origin.

8. Analyticity on circles for functions constant on lines

In the second example in Section 7 the function f is constant on each line passing through the origin, that is,

$$(8.1) \quad f(tz) = f(z) \quad (z \in \mathbb{C} \setminus \{0\}, t \in \mathbb{R} \setminus \{0\}).$$

In this section we look more closely at such functions.

THEOREM 8.1: *Suppose that f is a continuous function on $\mathbb{C} \setminus \{0\}$ that is a constant on each line passing through the origin, that is, f satisfies (8.1). If f*

extends holomorphically from one circle surrounding the origin then it extends holomorphically from every circle surrounding the origin. This happens if and only if there is a function g from the disc algebra such that $f(z) = g(z/\bar{z})$ ($z \in \mathbb{C} \setminus \{0\}$).

Proof: Suppose that f is a continuous function on $\mathbb{C} \setminus \{0\}$ that satisfies (8.1). Then there is a continuous function g on $b\Delta$ such that

$$(8.2) \quad f(z) = g(z/\bar{z}) \quad (z \in \mathbb{C} \setminus \{0\}).$$

Assume that f extends holomorphically from a circle $b\Delta(a, \rho)$ that surrounds the origin. By (8.1) we may assume that $\rho = 1$. If $a = 0$ then the function $\zeta \mapsto g(\zeta^2)$ ($\zeta \in b\Delta$) extends to a function in the disc algebra, which implies that g extends to a function from the disc algebra. Suppose that $a \neq 0$. Composing f with a rotation if necessary we may assume that $0 < a < 1$. By our assumption there is a function h from the disc algebra such that

$$h(\zeta) = g\left(\frac{a + \zeta}{a + \bar{\zeta}}\right) = g\left(\zeta \frac{a + \zeta}{1 + \zeta a}\right) \quad (\zeta \in b\Delta).$$

Clearly $\xi \mapsto h((\xi - t)/(1 - t\xi))$ ($\xi \in b\Delta$) extends to a function from the disc algebra for every $t, 0 \leq t < 1$. Put $t = (1 - \sqrt{1 - a^2})/a$. Then $0 < t < 1$ and

$$\frac{a + \frac{\xi - t}{1 - \xi t}}{1 + a \frac{\xi - t}{1 - \xi t}} = \frac{\xi + t}{1 + t\xi} \quad (\xi \in b\Delta),$$

which implies that

$$\xi \mapsto g\left(\frac{\xi - t}{1 - \xi t} \frac{\xi + t}{1 + \xi t}\right) = g\left(\frac{\xi^2 - t^2}{1 - t^2 \xi^2}\right) \quad (\xi \in b\Delta)$$

extends to a function from the disc algebra, which implies that

$$\xi \mapsto g((\xi - t^2)/(1 - t^2\xi)) \quad (\xi \in b\Delta)$$

extends to a function from the disc algebra. Consequently, g extends to a function in the disc algebra and so f is of the form (7.1). By the discussion following (7.1) it follows that the function f extends holomorphically from every circle that surrounds the origin.

9. More examples

Example 9.1: Let $0 < a < 1$ and let $\Phi(\zeta) = (a + \zeta)/|a + \zeta|$ ($\zeta \in b\Delta$). Then $\Phi: b\Delta \rightarrow b\Delta$ is diffeomorphism. Define

$$(9.1) \quad f(z) = \Phi^{-1}(z/|z|) \quad (z \in \mathbb{C} \setminus \{0\}).$$

The function f is continuous on $\mathbb{C} \setminus \{0\}$ and is constant on each ray emanating from the origin, that is,

$$(9.2) \quad f(tz) = f(z) \quad (z \in \mathbb{C} \setminus \{0\}, t > 0).$$

If an f satisfying (9.2) extends holomorphically from a circle $b\Delta(a, \rho)$ that surrounds the origin, then it extends holomorphically from $b\Delta(ta, t\rho)$ for every $t > 0$. So, when studying holomorphic extendibility from circles $b\Delta(a, \rho)$ we may, with no loss of generality, assume that $\rho = 1$.

Let f be as in (9.1). Since

$$f(a + \zeta) = \Phi^{-1}((a + \zeta)/|a + \zeta|) = \zeta \quad (\zeta \in b\Delta)$$

it follows that f extends holomorphically from $b\Delta(a, 1)$. By (9.2)

$$\frac{a + \zeta}{|a + \zeta|} = \sqrt{\frac{(a + \zeta)^2}{(a + \zeta)(a + \bar{\zeta})}} = \sqrt{\frac{a + \zeta}{a + \bar{\zeta}}} = \sqrt{\zeta \frac{a + \zeta}{1 + a\bar{\zeta}}} \quad (\zeta \in b\Delta);$$

it follows that

$$\zeta \mapsto f\left(\sqrt{\zeta \frac{a + \zeta}{1 + a\bar{\zeta}}}\right)$$

extends to a function from the disc algebra which happens if and only if

$$\zeta \mapsto f\left(\sqrt{M(\zeta) \frac{a + M(\zeta)}{1 + a\overline{M(\zeta)}}}\right)$$

extends to a function from the disc algebra for an automorphism M of Δ . In particular, if $M(\zeta) = (\zeta - a)/(1 - a\bar{\zeta})$ it follows that

$$\zeta \mapsto f\left(\sqrt{\zeta \frac{\zeta - a}{1 - a\bar{\zeta}}}\right) \quad (z \in b\Delta)$$

extends to a function from the disc algebra which is equivalent to the fact that f extends holomorphically from $b\Delta(-a, 1)$. It will follow from Theorem 10.1 that these two circles are the only circles of radius 1 from which f extends holomorphically.

Example 9.2: Let g be a function in the disc algebra which is not an even function. Let $f(z) = g(z/|z|)(z \in \mathbb{C} \setminus \{0\})$. Then f is continuous on $\mathbb{C} \setminus \{0\}$ and extends holomorphically from $b\Delta(0, 1)$. It will follow from Theorem 10.1 below that $b\Delta(0, 1)$ is the only circle of radius 1 from which f extends holomorphically.

10. Analyticity on circles for functions constant on rays

In both examples in Section 9 the function f satisfies (9.2), that is, f is constant on each ray emanating from the origin. In this section we look more closely at such functions.

Suppose that a continuous function f on $\mathbb{C} \setminus \{0\}$ satisfies (9.2). Assume that $0 \leq d < 1$ and that f extends holomorphically from $b\Delta(de^{i\alpha}, 1)$ for some $\alpha \in \mathbb{R}$. This means that $\zeta \mapsto f(e^{i\alpha}(d + \zeta))(\zeta \in b\Delta)$ extends to a function in the disc algebra. Since

$$e^{i\alpha}(d + \zeta)/|d + \zeta| = e^{i\alpha} \sqrt{\zeta(d + \zeta)/(1 + d\zeta)} \quad (\zeta \in b\Delta)$$

this happens if and only if

$$(10.1) \quad \begin{cases} f(e^{i\alpha} \sqrt{\zeta(d + \zeta)/(1 + d\zeta)}) = q(\zeta) & (\zeta \in b\Delta) \\ \text{where } q \text{ belongs to the disc algebra.} \end{cases}$$

In the case when $d = 0$ this implies that $\zeta \mapsto f(\zeta)(\zeta \in b\Delta)$ extends to a function from the disc algebra. Suppose that $d \neq 0$. Put

$$\zeta = \frac{\xi - t}{1 - t\xi} \quad \text{where } t = \frac{1 - \sqrt{1 - d^2}}{d}$$

to get

$$\zeta \frac{d + \zeta}{1 + d\zeta} = \frac{\xi^2 - t^2}{1 - t^2\xi^2},$$

so that (10.1) is equivalent to

$$(10.2) \quad \begin{cases} f\left(e^{i\alpha} \sqrt{\frac{\xi^2 - t^2}{1 - t^2\xi^2}}\right) = G(\xi) & (\xi \in b\Delta) \\ \text{where } G \text{ belongs to the disc algebra.} \end{cases}$$

In fact, $G(\xi) = q((\xi - t)/(1 - t\xi))(\xi \in b\Delta)$. Putting

$$Z = e^{i\alpha} \sqrt{(\xi^2 - t^2)/(1 - t^2\xi^2)}$$

we get

$$\xi^2 = \frac{(e^{-i\alpha} Z)^2 + t^2}{1 + t^2(e^{-i\alpha} Z)^2},$$

which implies that (10.2) is equivalent to

$$(10.3) \quad \begin{cases} f(Z) = G\left(e^{-i\alpha} \sqrt{\frac{Z^2 + e^{2i\alpha} t^2}{1 + e^{-2i\alpha} t^2 Z^2}}\right) & (Z \in b\Delta) \\ \text{where } G \text{ belongs to the disc algebra.} \end{cases}$$

THEOREM 10.1: *Let f be a continuous function on $\mathbb{C} \setminus \{0\}$ which satisfies (9.2), that is, f is constant on each ray emanating from the origin. Suppose that f extends holomorphically from $b\Delta(a, 1)$ and $b\Delta(b, 1)$ where $a, b \in \Delta$, $b \neq a$, $b \neq -a$. Then there is a function g in the disc algebra such that*

$$(10.4) \quad f(z) = g(z/|z|) \quad (z \in \mathbb{C} \setminus \{0\}).$$

Consequently, f satisfies (8.1), that is, f is constant on each line passing through the origin and extends holomorphically from each circle surrounding the origin.

Proof: Suppose that f extends holomorphically from $b\Delta(d_1 e^{i\alpha_1}, 1)$ and $b\Delta(d_2 e^{i\alpha_2}, 1)$ where $0 \leq d_i < 1$ ($i = 1, 2$), $d_2 e^{i\alpha_2} \neq d_1 e^{i\alpha_1}$, $d_2 e^{i\alpha_2} \neq -d_1 e^{i\alpha_1}$. It is enough to prove that f is an even function, for then the rest follows from Theorem 8.1.

Let $t_i = 0$ if $d_i = 0$ and $t_i = (1 - \sqrt{1 - d_i^2})/d_i$ if $d_i \neq 0$, $i = 1, 2$. Write $A_i = e^{2i\alpha_i} t_i^2$, $i = 1, 2$. By the discussion preceding Theorem 10.1 there are functions G_1, G_2 in the disc algebra such that

$$(10.5) \quad f(Z) = G_i \left(e^{-i\alpha_i} \sqrt{\frac{Z^2 + A_i}{1 + \overline{A_i} Z^2}} \right) \quad (Z \in b\Delta, i = 1, 2).$$

Write $W^2 = (Z^2 + A_1)/(1 + \overline{A_1} Z^2)$ so that $Z^2 = (W^2 - A_1)/(1 - \overline{A_1} W^2)$ ($W \in b\Delta$) and $(Z^2 + A_2)/(1 + \overline{A_2} Z^2) = (W^2 + C)/(1 + \overline{C} W^2)$ where $C = (A_2 - A_1)/(1 - A_1 A_2)$. Now (10.5) implies that

$$G_1 \left(e^{-i\alpha_1} \sqrt{\frac{Z^2 + A_1}{1 + \overline{A_1} Z^2}} \right) = G_2 \left(e^{-i\alpha_2} \sqrt{\frac{Z^2 + A_2}{1 + \overline{A_2} Z^2}} \right) \quad (Z \in b\Delta),$$

which implies that

$$(10.6) \quad G_1(e^{-i\alpha_1} W) = G_2 \left(e^{-i\alpha_2} \sqrt{\frac{W^2 + C}{1 + \overline{C} W^2}} \right) \quad (W \in b\Delta).$$

Since both G_1 and G_2 belong to the disc algebra it follows that the relation (10.6) continues holomorphically inside Δ , so (10.6) implies that either $C = 0$ or G_2 is an even function. Assume that $C = 0$. By (10.6) it follows that $A_1 = A_2$

and $e^{i\alpha_1} = e^{i\alpha_2}$. It follows that $d_2 e^{i\alpha_2} = \pm d_1 e^{i\alpha_1}$, which is impossible by the assumption. Thus, G_2 is an even function and consequently, by (10.5), f is an even function. This completes the proof.

Remark: Note that Theorem 10.1 implies that in Example 9.1 the circles $b\Delta(a, 1)$ and $b\Delta(-a, 1)$ are the only circles of radius one from which f extends holomorphically. Similarly, in Example 9.2, $b\Delta$ is the only circle of radius one from which f extends holomorphically.

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